Singular Value Decomposition

Basic Mathematical Tools for Imaging and Visualization (WT 2006)

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Singular Value Decomposition (SVD)

- For any given matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ there exists a decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{S}\mathbf{V}^\top$$

such that

  - $\mathbf{U}$ is an $(m \times m)$ orthogonal matrix
  - $\mathbf{S}$ is an $(m \times n)$ matrix with non-negative diagonal entries
  - $\mathbf{V}$ is an $(n \times n)$ orthogonal matrix

SVD - Visualized

$$\underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{\mathbf{A}(m\times n)} = \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}}_{\mathbf{U}(m\times m)} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \cdot & \\ & & \sigma_n \\ 0 & \cdot & 0 \\ 0 & \cdot & 0 \end{bmatrix}}_{\mathbf{S}(m\times n)} \underbrace{\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}}_{\mathbf{V}^\top(n\times n)}$$

- $\mathbf{A}$ is a $(m \times n)$ matrix
- $\mathbf{U}$ is a $(m \times m)$ orthogonal matrix
- $\mathbf{S}$ is a $(m \times n)$ matrix with non-negative diagonal entries
- $\mathbf{V}$ is a $(n \times n)$ orthogonal matrix

Singular Value Decomposition (SVD)

- The diagonal values of $\mathbf{S}$ are called Singular Values of $\mathbf{A}$
- The column vectors of $\mathbf{U}$ are the Left Singular Vectors of $\mathbf{A}$
- The column vectors of $\mathbf{V}$ are the Right Singular Vectors of $\mathbf{A}$.

Properties

- The SVD can be performed s.t. the diagonal values of $\mathbf{S}$ are descending i.e. $\sigma_1 \geq \sigma_2 \geq ... \geq \sigma_n \geq 0$.
- We will assume that the SVD is always performed in that way
- The diagonal values of $\mathbf{S}$ are the square roots of the Eigenvalues of $\mathbf{A}^\top\mathbf{A}$ and $\mathbf{A}\mathbf{A}^\top$
  (Hence the non-negativity of the entries of $\mathbf{S}$)

Some More Properties

- It holds for the left singular vectors $\mathbf{u}_i$:

$$\mathbf{A}^\top \mathbf{A}\mathbf{u}_i = {\sigma_i}^2 \mathbf{u}_i$$

- It holds for the right singular vectors $\mathbf{v}_i$:

$$\mathbf{A}\mathbf{A}^\top \mathbf{v}_i = {\sigma_i}^2 \mathbf{v}_i$$

- The left singular vectors $\mathbf{u}_i$ are eigenvectors of $\mathbf{A}^\top \mathbf{A}$
- The right singular vectors $\mathbf{v}_i$ are eigenvectors of $\mathbf{A}\mathbf{A}^\top$

Even More Properties

- SVD explicitly constructs orthonormal bases for the null-space and the range of a matrix
- columns of $\mathbf{U}$ corresponding to non-zero entries of $\mathbf{S}$ span the range
- columns of $\mathbf{V}$ corresponding to zero entries of $\mathbf{S}$ span the null-space

Properties Galore

- SVD allows a rank decision:

$$rank(\mathbf{A}) \text{ is the largest } r \text{ s.t. } \sigma_r > 0$$

- there are $m - r$ left singular vectors corresponding to the singular value 0.
- there are $n - r$ right singular vectors corresponding to the singular value 0.

Linear Optimization

SVD can be used for linear optimization by using the following property

- Let $\mathbf{v}_n$ be the right singular vector corresponding to σ_n (the smallest entry of $\mathbf{S}$)
- The product $\mathbf{Ax}$ with $\|\mathbf{x}\|_2 = 1$ has the minimal value for $\mathbf{x} = \mathbf{v}_n$.

Minimization by SVD

The minimizing property of the last right singular vector $\mathbf{v}_n$ can be used to solve the following minimization task

- Given the linear function $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, $f : \mathbf{R}^n \rightarrow \mathbf{R}^m$ to be minimized (in most applications $m >> n$)
- With the constraint that the solution $\mathbf{x}$ is not trivial $(\mathbf{x} \neq \mathbf{0})$ (We will assume that $\|\mathbf{x}\|_2 = 1$)

Minimization by SVD II

- The minimization problem is thus

$$\text{minimize} \quad \mathbf{Ax} \quad \text{s.t. } \|\mathbf{x}\|_2 = 1$$

- It can be shown that the solution is the right vector $\mathbf{x} = \mathbf{v}_n$ corresponding to the smallest singular value σ_n.

Proof

- Problem: minimize $\|\mathbf{Ax}\|_2$ subject to $\|\mathbf{x}\|_2 = 1$
- Because of the orthogonality of $\mathbf{U}$ and $\mathbf{V}$ we have
  - $\|\mathbf{Ax}\|_2 = \|\mathbf{USV}^\top\mathbf{x}\|_2 = \|\mathbf{SV}^\top\mathbf{x}\|_2$
  - $\|\mathbf{x}\|_2 = \|\mathbf{V}^\top\mathbf{x}\|_2$
- Hence have to minimize $\|\mathbf{SV}^\top\mathbf{x}\|_2$ subject to $\|\mathbf{V}^\top\mathbf{x}\|_2 = 1$
- With $\mathbf{V}^\top\mathbf{x} = \mathbf{y}$ we have: minimize $\|\mathbf{Sy}\|_2$ subject to $\|\mathbf{y}\|_2 = 1$
- Since $\mathbf{S}$ diagonal with descending entries we get $\mathbf{y} = [0, 0, ..., 0, 1]^\top$
- Since $\mathbf{V}^\top\mathbf{x} = \mathbf{y} \Leftrightarrow \mathbf{x} = \mathbf{Vy}$ we get $\mathbf{x} = \mathbf{v}_n$

What's next...?

So in order to solve a linear minimization problem by SVD we have to do two things

1. State it in the form

$$\text{minimize} \quad \mathbf{Ax} \quad \text{s.t.} \quad \|\mathbf{x}\|_2 = 1$$

2. Compute the SVD $\mathbf{A} = \mathbf{USV}^\top$ and take the last right vector $\mathbf{v}_n$ as the solution

Notes

- Minimization by SVD is extremely simple. The tricky part is to state the problem the right way.
- The error minimized by the SVD is called Algebraic Error or Algebraic Distance

SVD Black Box

- Only thing left is to show how an SVD of a matrix $\mathbf{A}$ can be computed
- But we won't do that!
- We'll use SVD as a black box
- Computation involves QR procedure and Householder reduction
- Original algorithm by Golub and Reinsch

Computation Properties

- Algorithm is extremely stable
- Computation time for SVD of a $(m \times n)$ matrix $\mathbf{A}$:
  - Computation of $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{S}$: $4m^2n + 8mn^2 + 9n^3$
  - Computation of $\mathbf{V}$ and $\mathbf{S}$: $4mn^2 + 8n^3$
- Keep in mind that in most cases $m >> n$

Implementation in Matlab

- Matlab has the command [U,S,V] = SVD(X)
- Attention: The command returns $\mathbf{V}$ and not $\mathbf{V}^\top$
- Hence it holds that $\mathbf{X} = \mathbf{U}\mathbf{S}\mathbf{V}^\top$

Summary

- Properties of the SVD
- Linear minimization using SVD

References

- Hartley and Zisserman. *Multiple View Geometry in Computer Vision*
- Walter Gander. *Fitting Data by Least Squares - Algorithms and Examples*
- Gander and Hrebicek. *Solving Problems in Scientific Computing using Maple and Matlab*
- Vachenauer. *Höhere Mathematik 1*
- Press et. al. *Numerical Recepies in C*
- Golub and van Loan. *Matrix Computations*