LAPLACE-BELTRAMI EIGENFUNCTIONS FOR DEFORMATION INVARIANT SHAPE REPRESENTATION

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Motivation

- Deformable shapes
  - Computer graphics
  - Shape modeling
  - Medical imaging
  - 3D face recognition

- Achieve deformation/pose invariant
  - Retrieval/matching
  - Correspondence
  - Segmentation
General approach

- Natural articulations
  - pair-wise geodesic distances change little
  - isometries – metric tensor stays same

- Deformation invariant embedding
  - Only metric properties are used
  - Produce an embedding of the surface into (higher dimensional) Euclidean space
  - The object and its deformations have the same embedding

- Segmentation, descriptor extraction, etc. uses this embedding – deformation invariance is achieved
Geodesics based embeddings

- Spectral embedding – MDS, Jain-Zhang
  - Pairwise geodesic distances between points
  - Flatten this structure – get embedding
  - Euclidean distance in embedding = geodesic dist.

- Successful:
  - classification, correspondence, segmentation

- Problems:
  - Geodesic distances are sensitive to local topology changes
  - A “short circuit” can affect a lot of geodesics
Our approach

- Construct an embedding
  - Geodesic distances are never used
  - Laplace-Beltrami eigenfunctions guide the construction

- Eigenfunctions have global nature
  - more stability to local changes

- Eigenfunctions are isometry invariant
  - Deformation invariant representation
Laplace-Beltrami

- Eigenvalues, eigenfunctions $\Delta \phi = \lambda \phi$
- Eigenvalues $\lambda_0 = 0 < \lambda_1 < \lambda_2 < \ldots < \lambda_i < \ldots$
- Eigenfunctions $\phi_i$:
  - Constitute an orthogonal basis
  - Bruno Levy: this basis is the one!
Global Point Signatures

- Given a point \( p \) on the surface we define

\[
GPS(p) = \left( \frac{1}{\sqrt{\lambda_1}} \phi_1(p), \frac{1}{\sqrt{\lambda_2}} \phi_2(p), \frac{1}{\sqrt{\lambda_3}} \phi_3(p), \ldots \right)
\]

- \( \phi_i(p) \) is the value of the eigenfunction at the point \( p \)
- Reason for square roots will be explained later
GPS embedding

- GPS can be considered as a mapping from the surface into infinite dimensional space.
- The image of this map will be called the **GPS embedding** of the surface.
- The infinite dimensional ambient space the **GPS domain**
Property 1: distinctness

- A surface without self-intersections is mapped into a surface without self-intersections
- In other words: distinct points have distinct images under the GPS.
Property 2: invariance

- GPS embedding is an isometry invariant.
  - Two isometric surfaces will have the same image under the GPS mapping
  - Same GPS embedding

- Reason:
  - Laplace-Beltrami operator is defined completely in terms of the metric tensor
  - LB is isometry invariant
  - LB eigenvalues and eigenfunctions of isometric surfaces coincide - their GPS embeddings also coincide
Property 3: reconstruction

- Given the GPS embedding and the eigenvalues, one can recover the surface *up to isometry*.
- Eigenvalues and eigenvectors of LB uniquely determine the metric tensor.
  
  This stems from completeness of eigenfunctions, which implies the knowledge of Laplace-Beltrami, from which one immediately recovers the metric tensor and so, the isometry class of the surface.
Property 4

- GPS embedding is absolute: it is not subject to rotations or translations of the ambient infinite-dimensional space.

- Compare with Geodesic MDS embedding
  - Determined only up to translations and rotations
  - There is no uniquely determined positional normalization relative to the embedding domain.
  - In order to compare two shapes, one still needs to find the appropriate rotations and translations to align the MDS embeddings of the shapes.
The GPS embedding is uniquely determined
- two isometric surfaces will have exactly the same GPS embedding
- except for reflections, because the signs of eigenfunctions are not fixed
- no rotation or translation in the ambient infinite dimensional space will be involved
- Example: the center of mass of the GPS embedding will automatically coincide with the origin
Property 5: meaningful distance

- The inner product and, thereby, the Euclidean distance in the GPS domain have a meaningful interpretation.
- Green’s function $G(x, x')$

$$G(x, x') = \sum_{i=1}^{\infty} \frac{\phi_i(x)\phi_i(x')}{\lambda_i}.$$

- The dot product in ambient space has meaning: $G(p, q) = GPS(p) \cdot GPS(q)$.
Discrete Setting

- Use Laplacian of Xu
- It is not symmetric
- We explain how to handle the non-symmetry
- Several novel remarks: complementary to “No Free Lunch”:
  - Wardetzky et al. prove that there is no discrete Laplacian that satisfies a set of requirements including symmetry
  - We show that one should not require a Laplacian to be symmetric
  - Also see “Symmetric Laplacian Considered Harmful”
Experiments

- Deformable shape classification
- G2 distributions
  - A variant of D2, but computed on the GPS embedding
  - Automatically deformation (isometry) invariant
Stability

- The global nature of eigenfunctions makes the G2 stable under local topol
Isometry invariance: dataset

- Yoshizawa et al.
Isometry invariance: MDS plot
Sample segmentation

- K-means clustering in the GPS, not
Problems

- Inability to deal with degenerate meshes
- Surfaces with boundaries impose appropriate boundary conditions.
- Two problems while working with eigenvalues and eigenvectors in general:
  - the signs of eigenvectors are undefined
  - two eigenvectors may be swapped
- Using D2 distributions indirectly addresses both of these issues.
- Further analysis is needed to clarify the consequences of these factors for shape processing when the GPS embedding is used directly
Acknowledgements

- Doctor Steve Novotny for not putting my fractured finger into a cast – this paper would not be possible
- Anonymous reviewers for their detailed and useful comments -- helped improve the paper immensely
- All models except the Dinopet and the sphere are from AIM@SHAPE Shape Repository; Deformations of Armadillo courtesy Shin Yoshizawa; the rest of the models are courtesy of INRIA

Thank You!